Boundaries of Levi-flat hypersurfaces: special hyperbolic points

Pierre Dolbeault

BOUNDARIES OF LEVI-FLAT HYPERSURFACES: SPECIAL HYPERBOLIC POINTS

PIERRE DOLBEAULT

Abstract. Let $S \subset \mathbb{C}^n$, $n \geq 3$ be a compact connected 2-codimensional submanifold having the following property: there exists a Levi-flat hypersurface whose boundary is S, possibly as a current. Our goal is to get examples of such S containing at least one special 1-hyperbolic point: sphere with two horns; elementary models and their gluing. The particular cases of graphs are also described.

1. Introduction

Let $S \subset \mathbb{C}^n$, be a compact connected 2-codimensional submanifold having the following property: there exists a Levi-flat hypersurface $M \subset \mathbb{C}^n \setminus S$ such that $dM = S$ (i.e. whose boundary is S, possibly as a current). The case $n = 2$ has been intensively studied since the beginning of the eighties, in particular by Bedford, Gaveau, Klingenberg; Shcherbina, Chirka, G. Tomassini, Slodkowski, Gromov, Eliashberg; it needs global conditions: S has to be contained in the boundary of a srictly pseudoconvex domain.

We consider the case $n \geq 3$; results on this case has been obtained since 2005 by Dolbeault, Tomassini and Zaitsev, local necessary conditions recalled in section 2 have to be satisfied by S, the singular CR points on S are supposed to be elliptic and the solution M is obtained in the sense of currents [DTZ05, DTZ10]. More recently a regular solution M has been obtained when S satisfies a supplementary global condition as in the case $n = 2$ [DTZ09], the singular CR points on S still supposed to be elliptic.

The problem we are interested in is to get examples of such S containing at least one special 1-hyperbolic point (section 2.4). The CR-orbits near a special 1-hyperbolic point are large and, assuming them compact, a careful examination has to be done (sections 2.6, 2.7). As a topological preliminary, we need a generalization of a theorem of Bishop on the difference of the numbers of special elliptic and 1-hyperbolic points (section 2.8); this result is a particular case of a theorem of Hon-Fei Lai [Lai72].

The first considered example is the sphere with two horns which has one special 1-hyperbolic point and three special elliptic points (section 3.4). Then we consider elementary models and their gluing to obtain more complicated examples (section 3.5). Results have been announced in [Dol08], and

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in more precise way in [Dol11]; the first aim of this paper is to give complete proofs. Finally, we recall in detail and extend the results of [DTZ09] on regularity of the solution when S is a graph satisfying a supplementary global condition, as in the case $n = 2$, to the case of existence of special 1-hyperbolic points, and to gluing of elemetary smooth models (section 4).

2. Preliminaries: local and global properties of the boundary

2.1. **Definitions.** A smooth, connected, CR submanifold $M \subset \mathbb{C}^n$ is called *minimal* at a point p if there does not exist a submanifold N of M of lower dimension through p such that $HN = HM|_N$. By a theorem of Sussman, all possible submanifolds N such that $HN = HM|_N$ contain, as germs at p, one of the minimal possible dimension, defining a so called CR *orbit* of p in M whose germ at p is uniquely determined.

Let S be a smooth compact connected oriented submanifold of dimension $2n-2$. S is said to be a *locally flat boundary* at a point p if it locally bounds a Levi-flat hypersurface near p. Assume that S is CR in a small enough neighborhood U of $p \in S$. If all CR orbits of S are 1-codimensional (which will appear as a necessary condition for our problem), the following two conditions are equivalent [DTZ05]:
(i) S is a locally flat boundary on U;
(ii) S is nowhere minimal on U.

2.2. **Complex points of S.** (i.e. singular CR points on S) [DTZ05].

At such a point $p \in S$, T_pS is a complex hyperplane in $T_p\mathbb{C}^n$. In suitable local holomorphic coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ vanishing at p, with $w = z_n$ and $z = (z_1, \dots, z_{n-1})$, S is locally given by the equation

$$(1) \qquad w = Q(z) + O(|z|^3), \quad Q(z) = \sum_{1 \le i,j \le n-1} (a_{ij} z_i z_j + b_{ij} z_i \overline{z}_j + c_{ij} \overline{z}_i \overline{z}_j)$$

S is said *flat* at a complex point $p \in S$ if $\sum b_{ij} z_i \overline{z}_j \in \lambda \mathbf{R}, \lambda \in \mathbb{C}$. We also say that *$p$ is flat*.

Let $S \subset \mathbb{C}^n$ be a locally flat boundary with a complex point p. Then p is flat.

By making the change of coordinates $(z, w) \mapsto (z, \lambda^{-1} w)$, we get $\sum b_{ij} z_i z_j \in \mathbb{R}$ for all z. By a change of coordinates $(z, w) \mapsto (z, w + \sum a'_{ij} z_i z_j)$ we can choose the holomorphic term in (1) to be the conjugate of the antiholomorphic one and so make the whole form Q real-valued.

We say that S is in a *flat normal form* at p if the coordinates (z, w) as in (1) are chosen such that $Q(z) \in \mathbf{R}$ for all $z \in \mathbb{C}^{n-1}$.

2.2.1. *Properties of Q.* Assume that S is in a flat normal form; then, the quadratic form Q is real valued. If Q is positive definite or negative definite, the point $p \in S$ is said to be *elliptic*; if the point $p \in S$ is not elliptic, and if Q is non degenerate, p is said to be *hyperbolic*. From section 2.4, we will only consider particular cases of the quadratic form Q.

2.3. **Elliptic points.**

2.3.1. *Properties of Q.*

Proposition 1. *([DTZ05, DTZ10]). Assume that $S \subset \mathbb{C}^n$, $(n \geq 3)$ is nowhere minimal at all its CR points and has an elliptic flat complex point p. Then there exists a neighborhood V of p such that $V \setminus \{p\}$ is foliated by compact real $(2n-3)$-dimensional CR orbits diffeomorphic to the sphere $\mathbf{S}^{2n-3}$ and there exists a smooth function ν, having the CR orbits as the level surfaces.*

Sketch of Proof. (see [DTZ10]). In the case of a quadric S_0 $(w = Q(z))$, the CR orbits are defined by $w_0 = Q(z)$, where w_0 is constant. Using (1), we approximate the tangent space to S by the tangent space to S_0 at a point with the same coordinate z; the same is done for the tangent spaces to the CR orbits on S and S_0; then we construct the global CR orbit on S through any given point close enough to p. $\square$

2.4. **Special flat complex points.** From [Bis65], for $n = 2$, in suitable local holomorphic coordinates centered at 0, $Q(z) = (z\overline{z} + \lambda Re\ z^2)$, $\lambda \geq 0$, under the notations of [BK91]; for $0 \leq \lambda < 1$, p is said to be *elliptic*, and for $1 < \lambda$, it is said to be *hyperbolic*. The parabolic case $\lambda = 1$, not generic, will be omitted [BK91]. When $n \geq 3$, the Bishop's reduction cannot be generalized.

We say that the flat complex point $p \in S$ is *special* if in convenient holomorphic coordinates centered at 0,

$$Q(z) = \sum_{j=1}^{n-1} (z_j \overline{z}_j + \lambda_j Re\ z_j^2), \quad , \lambda_j \geq 0 \tag{2}$$

Let $z_j = x_j + iy_j$, x_j, y_j real, $j = 1, \ldots, n-1$, then:

$$Q(z) = \textstyle\sum_{l=1}^{n-1} \left((1+\lambda_l)x_l^2 + (1-\lambda_l)y_l^2\right) + O(|z|^3). \tag{3}$$

A flat point $p \in S$ is said to be *special elliptic* if $0 \leq \lambda_j < 1$ for any j.

A flat point $p \in S$ is said to be *special k-hyperbolic* if $1 < \lambda_j$ for $j \in J \subset \{1, \ldots, n-1\}$ and $0 \leq \lambda_j < 1$ for $j \in \{1, \ldots, n-1\} \setminus J \neq \emptyset$, where k denotes the number of elements of J.

Special elliptic (resp. special k-hyperbolic) points are elliptic (resp. hyperbolic).

Special flat complex points

2.5. **Special hyperbolic points.** S being given by (1), let S_0 be the quadric of equation $w = Q(z)$.

Lemma 2. *Suppose that S_0 is flat at 0 and that 0 is a special k-hyperbolic point. Then, in a neighborhood of 0, and with the above local coordinates, S_0 is CR and nowhere minimal outside 0, and the CR orbits of S_0 are the $(2n-3)$-dimensional submanifolds given by $w = const. \neq 0$.*

Proof. The submanifolds $w = const. \neq 0$ have the same complex tangent space as S_0 and are of minimal dimension among submanifolds having this property, so they are CR orbits of codimension 1, and from the end of section 2.1, S_0 is nowhere minimal outside 0.

The section $w = 0$ of S_0 is a real quadratic cone Σ'_0 in $\mathbf{R}^{2n}$ whose vertex is 0 and, outside 0, it is a CR orbit Σ_0 in the neighborhood of 0. We will improperly call Σ'_0 a *singular CR orbit.* □

2.6. **Foliation by** CR**-orbits in the neighborhood of a special** 1**-hyperbolic point.** We first mimic and transpose the begining of the proof of Proposition 1, i.e. of 2.4.2. in ([DTZ05, DTZ09]).

2.6.1. *Local* 2*-codimensional submanifolds.* In order to use simple notations, we will assume $n = 3$.

In $\mathbb{C}^3$, consider the 4-dimensional submanifold S locally defined by the equation

$$w = \varphi(z) = Q(z) + O(|z|^3) \tag{1}$$

and the 4-dimensional submanifold S_0 of equation

$$w = Q(z) \tag{4}$$

with

$$Q = (\lambda_1 + 1)x_1^2 - (\lambda_1 - 1)y_1^2 + (1 + \lambda_2)x_2^2 + (1 - \lambda_2)y_2^2$$

having a special 1-hyperbolic point at 0, ($\lambda_1 > 1, 0 \leq \lambda_2 < 1$), and the cone Σ'_0 whose equation is: $Q = 0$. On S_0, a CR orbit is the 3-dimensional submanifold $\mathcal{K}_{w_0}$ whose equation is $w_0 = Q(z)$. If $w_0 > 0$, $\mathcal{K}_{w_0}$ does not cut the line $L = \{x_1 = x_2 = y_2 = 0\}$; if $w_0 < 0$, $\mathcal{K}_{w_0}$ cuts L at two points.

Lemma 3. $\Sigma_0 = \Sigma'_0 \setminus 0$ *has two connected components in a neighborhood of 0.*

Proof. The equation of $\Sigma'_0 \cap \{y_1 = 0\}$ is
$(\lambda_1 + 1)x_1^2 + (1 + \lambda_2)x_2^2 + (1 - \lambda_2)y_2^2 = 0$ whose only zero , in the neighborhood of 0, is $\{0\}$: the connected components are obtained for $y_1 > 0$ and $y_1 < 0$ respectively. □

Local 2-codimensional submanifolds

2.6.2. CR*-orbits.* By differentiating (1), we get for the tangent spaces the following asymptotics

$$T_{(z,\varphi(z))}S = T_{(z,Q(z))}S_0 + O(|z|^2), \quad z \in \mathbb{C}^2 \tag{5}$$

Here both $T_{(z,\varphi(z))}S$ and $T_{(z,Q(z))}S_0$ depend continuously on z near the origin.

Consider

(i) the hyperboloïd $H_- = \{Q = -1\}$, (then $Q(\frac{z}{(-Q(z))^{1/2}}) = -1$), and the projection:

$$\pi_- : \mathbb{C}^3 \setminus \{z = 0\} \to H_-, \quad (z, w) \mapsto \frac{z}{(-Q(z))^{1/2}},$$

(ii) for every $z \in H_-$, a real orthonormal basis $e_1(z), \ldots, e_6(z)$ of $\mathbb{C}^3 \cong \mathbb{R}^6$ such that

$$e_1(z), e_2(z) \in H_z H_-, \quad e_3(z) \in T_z H_-,$$

where HH_- is the complex tangent bundle to H_-.

Locally such a basis can be chosen continuously depending on z. For every $(z, w) \in \mathbb{C}^3 \setminus \{z = 0\}$, consider the basis $e_1(\pi_-(z, w)), \ldots, e_6(\pi_-(z, w))$. The unit vectors $e_1(\pi_-(z, w_0)), e_2(\pi_-(z, w_0)), e_3(\pi_-(z, w_0))$ are tangent to the CR orbit $\mathcal{K}_{w_0}$ in (z, w_0) for $w_0 < 0$. Then, from (5), we have:

$$H_{(z,\varphi(z))}S = H_{(z,Q(z))}S_0 + O(|z|^2), \quad z \neq 0, \ \ z \to 0. \tag{6}$$

As in [DTZ10], in the neighborhood of 0, denote by $E(q), q \in S\setminus\{0\}, w < 0$ the tangent space to the local CR orbit $\mathcal{K}$ on S through q, and by $E_0(q_0), q_0 \in S_0 \setminus \{0\}, w < 0$ the analogous object for S_0. We have :

$$E(z, \varphi(z)) = E_0(z, Q(z)) + O(|z|^2), \ \ z \neq 0, \ \ z \to 0 \tag{7}$$

Given $\underline{q} \in S$, by integration of $E(q)$, $q \in S$, we get, locally, the CR orbit (the leaf), on S through $\underline{q}$; given $\underline{q_0} \in S_0$, by integration of $E_0(q_0)$, $q_0 \in S_0$, we get, locally, the CR orbit (the leaf), on S_0 through $\underline{q_0}$ (theorem of Sussman). On S_0, a leaf is the 3-dimensional submanifold $\mathcal{K}_{\underline{q_0}} = \mathcal{K}_{w_0} = \mathcal{K}_0$ whose equation is $w_0 = Q(z)$, with $\underline{q} = (z_0, w_0 = Q(z_0))$. $d\pi_-$ projects each $E_0(q)$, $q \in S_0, w < 0$, bijectively onto $T_{\pi(q)}H_-$, then $\pi_-|_{\mathcal{K}_0}$ is a diffeomorphism onto H_-; this implies, from (7), that, in a suitable neighborhood of the origin, the restriction of π_- to each local CR orbit of S is a local diffeomorphism.

We have: $\varphi(z) = Q(z) + \Phi(z)$ with $\Phi(z) = O(|z|^3)$.

2.6.3. *Behaviour of local CR orbits.* Follow the construction of $E(z, \varphi(z))$; compare with $E_0(z, Q(z))$. We know the integral manifold, the orbit of $E_0(z, Q(z))$; deduce an evaluation of the integral manifold $\mathcal{K}$ of $E(z, \varphi(z))$.

Lemma 4. *Under the above hypotheses, the local orbit Σ corresponding to Σ_0 has two connected components in the neighborhood of* 0.

Proof. Using the real coordinates, as for Lemma 3, $\Sigma' \cap \{y_1 = 0\}$. Locally, the connected components are obtained for $y_1 > 0$ and $y_1 < 0$ respectively, from formula (1). ☐

We will improperly call $\Sigma' = \overline{\Sigma}$ a *singular CR orbit* and a *singular leaf of the foliation.*

We intend to prove: 1) $\mathcal{K}$ does not cross the singular leaf through 0;

2) the only separatrix is the singular leaf through 0.

From the orbit $\mathcal{K}_0$, construct the differential equation defining it, and using (7), construct the differential equation defining $\mathcal{K}$.

In $\mathbb{C}^3$, we use the notations: $x = x_1, y = y_1, u = x_2, v = y_2$; it suffices to consider the particular case: $Q = 3x^2 - y^2 + u^2 + v^2$. On S_0, the orbit $\mathcal{K}_0$ issued from the point $(c, 0, 0, 0)$ is defined by: $3x^2 - y^2 - u^2 + v^2 = 3c^2$, i.e., for $x \geq 0$, $x = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2 + 3c^2)^{\frac{1}{2}} = A(y, u, v)$; the local coordinates on the orbit are (y, u, v). $\mathcal{K}_0$ satisfies the differential equation: $dx = dA$. From (9), the orbit $\mathcal{K}$, issued from $(c, 0, 0, 0)$, satisfies $dx = dA + \Psi$ with $\Psi(y, u, v; c) = O(|z|^2)$; hence $\Psi = d\Phi$, then $x = A + \Phi$, with $\Phi = O(|z|^3)$. More explicitly, $\mathcal{K}$ is defined by:

$$x = x_{\mathcal{K},c} = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2 + 3c^2)^{\frac{1}{2}} + \Phi(y, u, v; c), \;\; \Phi(y, u, v; c) = O(|z|^3)$$

The cone Σ'_0 whose equation is: $Q = 0$ is a separatrix for the orbits $\mathcal{K}_0$. The corresponding object $\Sigma' = \{\varphi(z) = 0\}$ for S has the singular point 0 and for $x > 0, y > 0,\ u > 0, v > 0$ is defined by the differential equation $dx = d(A + \Phi)$, with $c = 0$, i.e. the local equation of Σ' is

$$x = x_{\mathcal{K},0} = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2)^{\frac{1}{2}} + \Phi(y, u, v; 0), \;\; \Phi(y, u, v; 0) = O(|z|^3)$$

For given (y, u, v), $x_{\mathcal{K},c} - x_{\mathcal{K},0} = x_{\mathcal{K}_0,c} - x_{\mathcal{K}_0,0} + \Phi(y, u, v; c) - \Phi(y, u, v; 0)$. But $x_{\mathcal{K}_0,c} - x_{\mathcal{K}_0,0} = O(1)$ and $\Phi(y, u, v; c) - \Phi(y, u, v; 0) = O(|z|^3)$.

As a consequence, for $x > 0, y > 0, u > 0, v > 0$, locally, Σ' is a separatrix for the orbits $\mathcal{K}$, and the only one. Same result for $x < 0$.

2.6.4. What has been done from the hyperboloïd $H_- = \{Q = -1\}$ can be repeated from the hyperboloïd $H_+ = \{Q = 1\}$.

As at the beginning of the section 2.6.2, we consider

(i) the hyperboloïd $H_+\{Q = 1\}$ and the projection:

$$\pi_+ : \mathbb{C}^3 \setminus \{z = 0\} \to H_+, \quad (z, w) \mapsto \frac{z}{(Q(z))^{1/2}},$$

(ii) for every $z \in H_+$, a real orthonormal basis $e_1(z), \dots, e_6(z)$ of $\mathbb{C}^3 \cong \mathbb{R}^6$ such that

$$e_1(z), e_2(z) \in H_z H_+, \quad e_3(z) \in T_z H_+,$$

where HH_+ is the complex tangent bundle to H_+.

2.6.5.

Lemma 5. *Given φ, there exists $R > 0$ such that, in $B(0, R) \cap \{x > 0, y > 0, u > 0, v > 0\} \subset \mathbb{C}^2$, the CR orbits $\mathcal{K}$ have Σ' as unique separatrix.*

Proof. When c tends to zero, , $x_{\mathcal{K},c} - x_{\mathcal{K},0} = x_{\mathcal{K}_0,c} - x_{\mathcal{K}_0,0} = O(|z|)$, $\Phi(y,u,v;c) - \Phi(y,u,v;0) = O(|z|^3)$. For $\varphi(z) = Q(z) + \Phi(z)$ with $\Phi(z) = O(|z|^3)$ given, in (9), $E(z,\varphi(z)) - E_0(z,Q(z)) = O(|z|^2)$ and $\Phi(y,u,v;c) - \Phi(y,u,v;0) = O(|z|^3)$ are also given. Then there exists R such that, for $|z| < R$, $x_{\mathcal{K},c} - x_{\mathcal{K},0} > 0$. □

2.7. CR–orbits near a subvariety containing a special 1-hyperbolic point.

2.7.1. In the section 2.7, we will impose conditions on S and give a local property in the neighborhood of a compact $(2n-3)$-subvariety of S.

Assume that $S \subset \mathbb{C}^n$ $(n \geq 3)$, is a locally closed $(2n-2)$-submanifold, nowhere minimal at all its CR points, which has a unique 1-hyperbolic flat complex point p, and such that:

(i) Σ being the orbit whose closure Σ' contains p, then Σ' is compact.

Let $q \in S$, $q \neq p$; then, in a neighborhood U of q disjoint from p, S is CR, CR-dim $S = n-2$, S is non minimal and Σ is 1-codimensional. To show that the CR orbits contitute a foliation on S whose separatrix is Σ': this is true in U since $\Sigma \cap U$ is a leaf. Moreover, let U_0 the ball $B(0,R)$ centered in $p = 0$ in Lemma 5, if $U \cap U_0 \neq \emptyset$, the leaves in U glue with the leaves in U_0 on $U \cap U_0$. Since Σ' is compact, there exists a finite number of points $q_j \in \Sigma'$, $j = 0, 1, \ldots, J$, and open neighborhoods U_j, as above, such that $(U_j)_{j=0}^J$ is an open covering of Σ'. Moreover the leaves on U_j glue respectively with the leaves on U_k if $U_j \cap U_h \neq \emptyset$.

2.7.2.

Proposition 6. *Assume that $S \subset \mathbb{C}^n$ $(n \geq 3)$, is a locally closed $(2n-2)$-submanifold, nowhere minimal at all its CR points, which has a unique special 1-hyperbolic flat complex point p, and such that:*

(i) Σ being the orbit whose closure Σ' contains p, then Σ' is compact;

(ii) Σ has two connected components σ_1, σ_2, whose closures are homeomorphic to spheres of dimension $2n-3$.

Then, there exists a neighborhood V of Σ' such that $V \setminus \Sigma'$ is foliated by compact real $(2n-3)$-dimensional CR orbits whose equation, in a neighborhood of p is (3), and, the $w(= x_n)$-axis being assumed to be vertical, each orbit is diffeomorphic to

the sphere $\mathbf{S}^{2n-3}$ above Σ',

the union of two spheres $\mathbf{S}^{2n-3}$ under Σ',

and there exists a smooth function ν, having the CR orbits as the level surfaces.

Proof. From subsection 2.7.1 and the following remark:

When x_n tends to 0, the orbits tends to Σ', and because of the geometry of the orbits near p, they are diffeomorphic to a sphere above Σ', and to the union of two spheres under Σ'. The existence of ν is proved as in Proposition 1, namely, consider a smooth curve $\gamma : [0, \varepsilon) \to S$ such that

$\gamma(0) = q$, where q is a point of Σ close to p, and γ is a diffeomorphism onto its image $\Gamma = \gamma([0, \varepsilon))$. Let $\nu = \gamma^{-1}$ on the image of γ, then, close enough to q, every CR orbit cuts Γ at a unique point $q(t)$, $t \in [0, \varepsilon)$. Hence there is a unique extension of ν from $\gamma([0, \varepsilon))$ to $V \setminus p$ where V is a neighborhood of Σ' having CR orbits as its level surfaces. ν being smooth away from p, it is smooth on the orbit Σ and, if we set $\nu(p) = \nu(q) = 0$, ν is smooth on a neighborhood of $\Sigma \cup \{p\} = \Sigma'$.

□

2.8. **Geometry of the complex points of S.** The results of section 2.8 are particular cases of theorems of H-F Lai [Lai72], that I learnt from F. Forstneric in July 2011.

In [BK91] E. Bedford & W. Klingenberg cite the following theorem of E. Bishop [Bis65][section 4, p.15]: *On a 2-sphere embedded in $\mathbb{C}^2$, the difference between the numbers of elliptic points and of hyperbolic points is the Euler-Poincaré characteristic, i.e.* 2. For the proof, Bishop uses a theorem of ([CS 51], section 4).

We extend the result for $n \geq 3$ and give proofs which are essentially the same than in the general case of [Lai72, Lai74] but simpler.

2.8.1. Let S be a smooth compact connected oriented submanifold of dimension $2n-2$. Let G be the manifold of the oriented real linear $(2n-2)$-subspaces of $\mathbb{C}^n$. The submanifold S of $\mathbb{C}^n$ has a given orientation which defines an orientation $o(p)$ of the tangent space to S at any point $p \in S$. By mapping each point of S into its oriented tangent space, we get a smooth Gauss map

$$t : S \to G$$

Denote $-t(p)$ the tangent space to S at p with opposite orientation $-o(p)$.

2.8.2. *Properties of G.* (a) dim $G = 2(2n-2)$.

Proof. G is a two-fold covering of the Grassmannian $M_{m,k}$, of the linear k-subspaces of $\mathbb{R}^m$ [Ste99][Part, section 7.9], for $m = 2n$ and $k = 2n-2$; they have the same dimension. We have:

$$M_{m,k} \cong O_m / O_k \times O_{m-k}$$

But dim $O_k = \dfrac{1}{2}k(k-1)$, hence dim $M_{m,k} = \dfrac{1}{2}\Big(m(m-1) - k(k-1) - (m-k)(m-k-1)\Big) = k(m-k)$.

(b) G has the complex structure of a smooth quadric of complex dimension $(2n-2)$ of $\mathbb{C}P^{2n-1}$ L74, [Pol08].

(c) There exists a canonical isomorphism $h : G \to \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1}$.

(d) Homology of G (cf [Pol08]): Let S_1, S_2 be generators of $H_{2n-2}(G, \mathbb{Z})$; we assume that S_1 and S_2 are fundamental cycles of complex projective subspaces of complex dimension $(n-1)$ of the complex quadric G. We also denote S_1, S_2 the ordered two factors $\mathbb{C}P^{n-1}$, so that $h : G \to S_1 \times S_2$.

□

2.8.3.

Proposition 7. *For $n \geq 2$, in general, S has isolated complex points.*

Proof. Let $\pi \in G$ be a complex hyperplane of $\mathbb{C}^n$ whose orientation is induced by its complex structure; the set of such π is $H = G^{\mathbb{C}}_{n-1,n} = \mathbb{C}\mathrm{P}^{n-1*} \subset G$, as real submanifold. If p is a complex point of S, then $t(p) \in H$ or $-t(p) \in H$. The set of complex points of S is the inverse image by t of the intersections $t(S) \cap H$ and $-t(S) \cap H$ in G. Since dim $t(S) = 2n-2$, dim $H = 2(n-1)$, dim $G = 2(2n-2)$, the intersection is 0-dimensional, in general. □

2.8.4. Denoting also S, the fundamental cycle of the submanifold S and t_* the homomorphism defined by t, we have:

$$t_*(S) \sim u_1 S_1 + u_2 S_2$$

where $\sim$ means *homologous to.*

2.8.5.

Lemma 8 (proved for $n = 2$ in [CS51])**.** *With the above notations, we have: $u_1 = u_2$; $u_1 + u_2 = \chi(S)$, Euler-Poincaré characteristic of S.*

The proof for $n = 2$ works for any $n \geq 3$, namely:

Let G' be the manifold of the oriented real linear 2-subspaces of $\mathbb{C}^n$. Let $\alpha : G \to G'$ map each oriented $2(n-1)$-subspace R onto its normal 2-subspace R' oriented so that R, R' determine the orientation of $\mathbb{C}^n$. α is a canonical isomorphism. Let $n : S \to G'$ the map defined by taking oriented normal planes; then: $n = \alpha t$ and $t = \alpha^{-1} n$, hence the mapping $h\alpha h^{-1} : S_1 \times S_2 \to S_1 \times S_2$. Let (x, y) be a point of $S_1 \times S_2$, then (†) $h\alpha h^{-1}(x, y) = (x, -y)$.

Over G, there is a bundle V of spheres obtained by considering as fiber over a real oriented linear $(2n-2)$-subspace of $\mathbb{C}^n$ through 0 the unit sphere $\mathbf{S}^{2n-3}$ of this subspace. Let Ω be the characteristic class of V, and let Ω_t, Ω_n denote the characteristic classes of the tangent and normal bundles of S. Then $t^*\Omega = \Omega_t$, $n^*\Omega = \Omega_n$.

V is the Stiefel manifold of ordered pairs of orthogonal unit vectors through in $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Let $f : V \to G$ the projection.

From the Gysin sequence, we see that the kernel of $f^* : H^{2n-2}(G) \to H^{2n-2}(V)$ is generated by Ω. To find the kernel of f^*, we determine the morphism $f_* : H_{2n-2}(V) \to H_{2n-2}(G)$. A generating $2n-2)$-cycle of in V is $S^2 \times e$ where $S^2 \cong \mathbb{C}P^{n-1}$ and e is a point. Let z be any point of S^2, then from (†), we have

$$hf(z, e) = (z, -z)$$

Therefore, we see that $f_*(S^2 \times e) = S_1 - S_2$. Then, the kernel of f^* is $\mathbb{Z}$-generated by $S_1^* + S_2^*$.

With convenient orientation for the fibre of the bundle V, we get: $\Omega = S_1^* + S_2^*$. For convenient orientation of S, we get $\Omega_t.S = \chi_S =$ Euler characteristic of S. We have

$$\Omega_t = t^*(S_1^* + S_2^*) = t^*S_1^* + t^*S_2^*$$

$$\Omega_n = n^*(S_1^* + S_2^*) = t^*\alpha^*(S_1^* + S_2^*) = t^*(S_1^* - S_2^*) = t^*S_1^* - t^*S_2^*$$

Since $\Omega_n = 0$, we get:

$$(t^*S_1^*).S = (t^*S_2^*).S = \frac{1}{2}\chi_S$$

2.8.6. *Local intersection numbers of H and $t(S)$ when all complex points are flat and special.* H is a complex linear $(n-1)$-subspace of G, then is homologous to one of the S_j, $j = 1, 2$, say S_2 when G has its structure of complex quadric. The intersection number of H and S_1 is 1 and the intersection number of H and S_2 is 0. So, the intersection number of H and $u_1S_1 + u_2S_2$ is u_1.
In the neighborhood of a complex point 0, S is defined by equation (1), with $w = z_n$ and

$$(1') \qquad Q(z) = \sum_{j=1}^{n-1} \mu_j(z_j\overline{z}_j + \lambda_j \mathcal{R}e\ z_j^2), \quad \mu_j > 0, \lambda_j \geq 0$$

Let $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n$, with real x_l. Let e_l the unit vector of the x_l axis, $l = 1, \ldots, 2n$.

For simplicity assume $n = 3$: $Q(z) = \mu_1(z_1\overline{z}_1 + \lambda_1\mathcal{R}e\ z_1^2) + \mu_2(z_2\overline{z}_2 + \lambda_2\mathcal{R}e\ z_2^2)$, with $\mu_1 = \mu_2 = 1$.

Then, up to higher order terms, S is defined by:

$z_1 = x_1 + ix_2$; $z_2 = x_3 + ix_4$; $z_3 = (1+\lambda_1)x_1^2 + (1-\lambda_1)x_2^2 + (1+\lambda_2)x_3^2 + (1-\lambda_2)x_4^2$.

In the neighborhood of 0, the tangent space to S is defined by the four independent vectors

$$\nu_1 = e_1 + 2(1+\lambda_1)x_1\, e_5; \quad \nu_2 = e_2 + 2(1-\lambda_1)x_2\, e_5; \quad \nu_3 = e_3 + 2(1+\lambda_2)x_3\, e_5;$$
$$\nu_4 = e_4 + 2(1-\lambda_2)x_4\, e_5$$

Then, if 0 is special elliptic or special k-hyperbolic with k even, the tangent plane at 0 has the same orientation; if 0 is special elliptic or special k-hyperbolic with k odd the tangent space has opposite orientation.

2.8.7.

Proposition 9 (known for $n = 2$ [Bis65], *here for* $n \geq 3$)**.** *Let S be a smooth, oriented, compact, 2-codimensional, real submanifold of $\mathbb{C}^n$ whose all complex points are flat and special elliptic or special 1-hyperbolic. Then, on S, $\sharp$ (special elliptic points) - $\sharp$ (special* 1*-hyperbolic points* $= \chi(S)$. *If S is a sphere, this number is 2.*

Proof. Let $p \in S$ be a complex point and π be the tangent hyperplane to S at π. Assume that

(**) *the orientation of S induces, on π, the orientation given by its complex structure*,

then $\pi \in H$.

If p is elliptic, the intersection number of H and $t(S)$ is 1; if p is 1-hyperbolic, the intersection number of H and $t(S)$ is -1 at p.

From the beginning of section 2.8.6, the sum of the intersection numbers of H and $t(S)$ at complex points p satisfying (**) is u_1. Reversing the condition (**), and using Lemma 8, we get the Proposition. ☐

3. Particular cases: horned sphere; elementary models and their gluing

3.1. We recall the following Harvey-Lawson theorem with real parameter to be used later.

3.1.1. Let $E \cong \mathbf{R} \times \mathbb{C}^{n-1}$, and $k : \mathbf{R} \times \mathbb{C}^{n-1} \to \mathbf{R}$ be the projection. Let $N \subset E$ be a compact, (oriented) CR subvariety of $\mathbb{C}^{n+1}$ of real dimension $2n-2$ and CR dimension $n-2$, $(n \geq 3)$, of class C^∞, with negligible singularities (i.e. there exists a closed subset $\tau \subset N$ of $(2n-2)$-dimensional Hausdorff measure 0 such that $N \setminus \tau$ is a CR submanifold). Let τ' be the set of all points $z \in N$ such that either $z \in \tau$ or $z \in N \setminus \tau$ and N is not transversal to the complex hyperplane $k^{-1}(k(z))$ at z. Assume that N, as a current of integration, is d-closed and satisfies:

(H) there exists a closed subset $L \subset \mathbb{R}_{x_1}$ with $H^1(L) = 0$ such that for every $x \in k(N) \setminus L$, the fiber $k^{-1}(x) \cap N$ is connected and does not intersect τ'.

3.1.2.

Theorem 10 ([DTZ10] (see also [DTZ05]))**.** *Let N satisfy* (H) *with L chosen accordingly. Then, there exists, in $E' = E \setminus k^{-1}(L)$, a unique C^∞ Levi-flat $(2n-1)$-subvariety M with negligible singularities in $E' \setminus N$, foliated by complex $(n-1)$-subvarieties, with the properties that M simply (or trivially) extends to E' as a $(2n-1)$-current (still denoted M) such that $dM = N$ in E'.1 The leaves are the sections by the hyperplanes $E_{x_1^0}$, $x_1^0 \in k(N) \setminus L$, and are the solutions of the "Harvey-Lawson problem" for finding a holomorphic subvariety in $E_{x_1^0} \cong \mathbb{C}^n$ with prescribed boundary $N \cap E_{x_1^0}$.*

3.1.3.

Remark 11. *Theorem 10 is valid in the space $E \cap \{\alpha_1 < x_1 < \alpha_2\}$, with the corresponding condition (H). Moreover, since N is compact, for convenient coordinate x_1, we can assume $x_1 \in [0,1]$.*

3.2. *To solve the boundary problem by Levi-flat hypersurfaces, S has to satisfy necessary and sufficient local conditions. A way to prove that these conditions can occur is to construct an example for which the solution is obvious.*

3.3. Sphere with one special 1-hyperbolic point (sphere with two horns): Example.

3.3.1. In $\mathbb{C}^3$, let (z_j), $j = 1, 2, 3$, be the complex coordinates and $z_j = x_j + iy_j$. In $\mathbf{R}^6 \cong \mathbb{C}^3$, consider the 4-dimensional subvariety (with negligible singularities) S defined by:
$y_3 = 0$
$0 \leq x_3 \leq 1$; $x_3(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 - 1) + (1 - x_3)(x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2) = 0$
$-1 \leq x_3 \leq 0$; $x_3 = x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2$

The singular set of S is the 3-dimensional section $x_3 = 0$ along which the tangent space is not everywhere (uniquely) defined. S being in the real hyperplane $\{y_3 = 0\}$, the complex tangent spaces to S are $\{x_3 = x^0\}$ for convenient x^0.

3.3.2. The tangent space to the hypersurface $f(x_1, y_1, x_2, y_2, x_3) = 0$ in $\mathbb{R}^5$ is

$$X_1 f'_{x_1} + Y_1 f'_{y_1} + X_2 f'_{x_2} + Y_2 f'_{y_2} + X_3 f'_{x_3} = 0,$$

Then, the tangent space to S in the hyperplane $\{y_3 = 0\}$ is:
for $0 \leq x_3$,

$$\begin{aligned} &2x_1[x_3 + 2(1 - x_3)(x_1^2 + 2)]X_1 + 2y_1[x_3 + 2(1 - x_3)(y_1^2 - 1)]Y_1 \\ &\quad + 2x_2[x_3 + (1 - x_3)(2x_2^2 + 1)]X_2 + 2y_2[x_3 + (1 - x_3)(2y_2^2 + 1)]Y_2 \\ &\qquad + [(x_1^2 + y_1^2 + x_2^2 + y_2^2 + 3x_3^2 - 1) \\ &\qquad\quad - (x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2)]X_3 = 0; \end{aligned}$$

for $x_3 \leq 0$,

$$4(x_1^2 + 2)x_1 X_1 + 4(y_1^2 - 1)y_1 Y_1 + 2(2x_2^2 + 1)x_2 X_2 + 2(2y_2^2 + 1)y_2 Y_2 - X_3 = 0.$$

3.3.3. The complex points of S are defined by the vanishing of the coefficients of X_j, j=1,2,3,4 in the equation of the tangent spaces
for $0 \leq x_3 \leq 1$,
$x_1[x_3 + 2(1 - x_3)(x_1^2 + 2)] = 0,$
$y_1[x_3 + 2(1 - x_3)(y_1^2 - 1)] = 0,$
$x_2[x_3 + (1 - x_3)(2x_2^2 + 1)] = 0,$
$y_2[x_3 + (1 - x_3)(2y_2^2 + 1)] = 0.$

We have the solutions

h: $x_j = 0, y_j = 0$, $(j = 1, 2)$, $x_3 = 0$;
e_3: $x_j = 0, y_j = 0$, $(j = 1, 2)$, $x_3 = 1$.
for $x_3 \leq 0$,

$(x_1^2 + 2)x_1 = 0$,
$(y_1^2 - 1)y_1 = 0$,
$(2x_2^2 + 1)x_2 = 0$,
$(2y_2^2 + 1)y_2 = 0$.

We have the solutions

h: $x_j = 0, y_j = 0$, $(j = 1, 2)$, $x_3 = 0$;
e_1, e_2: $x_1 = 0, y_1 = \pm 1, x_2 = 0, y_2 = 0, x_3 = -1$.

Remark that the tangent space to S at h is well defined. Moreover, the set S will be smoothed along its section by the hyperplane $\{x_3 = 0\}$ by a small deformation leaving h unchanged. *In the following S will denote this smooth submanifold.*

3.3.4.

Lemma 12. *The points e_1, e_2, e_3 are special elliptic; the point h is special $\{1\}$-hyperbolic.*

Proof. Point e_3: Let $x_3' = 1 - x_3$, then the equation of S in the neighborhood of e_3 is:

$(1 - x_3')(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3'^2 - 2x_3') - x_3'(x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2) = 0$, *i.e.*

$2x_3' = x_1^2 + y_1^2 + x_2^2 + y_2^2) + O(|z|^3)$, or $w = z\overline{z} + O(|z|^3)$

then e_3 is special elliptic.

Points e_1, e_2: Let $y_1' = y_1 \pm 1$, $x_3' = x_3 + 1$, then the equation of S in the neighborhood of e_1, e_2 is:

$x_3' - 1 = x_1^4 + (y_1' \mp 1)^4 + x_2^4 + y_2^4 + 4x_1^2 - 2(y_1' \mp 1)^2 + x_2^2 + y_2^2$
$= x_1^4 + y_1'^4 \mp 4y_1'^3 + 6y_1'^2 \mp 4y_1' + 1 + x_2^4 + y_2^4 + 4x_1^2 - 2(y_1' \mp 1)^2 + x_2^2 + y_2^2$,

then

$x_3' = x_1^4 + y_1'^4 \mp 4y_1'^3 + 4y_1'^2 + x_2^4 + y_2^4 + 4x_1^2 + x_2^2 + y_2^2$, *i.e.*
$x_3' = 4x_1^2 + 4y_1'^2 + x_2^2 + y_2^2 + O(|z|^3)$, or $w = 4z_1\overline{z}_1 + z_2\overline{z}_2$,

then e_1, e_2 are special elliptic.

Point h: The equation of S in the neighborhood of h is:
for $x_3 \geq 0$,

$$x_3(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 - 1) + (1 - x_3)(x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2) = 0$$

for $x_3 \leq 0$,

$x_3 = x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2$, *i.e.*

$x_3 = 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 + O(|z|^3)$, in both cases, up to the third order terms, *i.e.*: $w = z_1\overline{z}_1 + z_2\overline{z}_2 + 3\mathcal{R}e\ z_1^2$,
then h is special $\{1\}$-hyperbolic. $\square$

3.3.5. *Section* $\Sigma' = S \cap \{x_3 = 0\}$. Up to a small smooth deformation, its equation is:

$x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 = 0$, in $\{x_3 = 0\}$.
The tangent cone to Σ' at 0 is: $4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 = 0$.

Locally, the section of S by the coordinate 3-space
x_1, y_1, x_3 is: $x_3 = 4x_1^2 - 2y_1^2 + O(|z|^3)$
x_2, y_2, x_3 is: $x_3 = x_2^2 + y_2^2 + O(|z|^3)$

3.3.1'. *Shape of* $\Sigma' = S \cap \{x_3 = 0\}$ *in the neighborhood of the origin* 0 *of* $\mathbb{C}^3$.

Lemma 13. *Under the above hypotheses and notations,*
(*i*) $\Sigma = \Sigma' \setminus 0$ *has two connected components* σ_1, σ_2.
(*ii*) *The closures of the three connected components of* $S \setminus \Sigma'$ *are submanifolds with boundaries and corners.*

Proof. (*i*) The only singular point of Σ' is 0. We work in the ball $B(0, A)$ of $\mathbb{C}^2$ (x_1, y_1, x_2, y_2) for small A and in the 3-space $\pi_\lambda = \{y_2 = \lambda x_2\}$, $\lambda \in \mathbb{R}$. For λ fixed, $\pi_\lambda \cong \mathbb{R}^3(x_1, y_1, x_2)$, and $\Sigma' \cap \pi_\lambda$ is the cone of equation $4x_1^2 - 2y_1^2 + (1 + \lambda^2)x_2^2 + O(|z|^3) = 0$ with vertex 0 and basis in the plane $x_2 = x_2^0$ the hyperboloid H_λ of equation $4x_1^2 - 2y_1^2 + (1+\lambda^2)x_2^{02} + O(|z|^3) = 0$; the curves H_λ have no common point outside 0. So, when λ varies, the surfaces $\Sigma' \cap \pi_\lambda$ are disjoint outside 0. The set Σ' is clearly connected; $\Sigma' \cap \{y_1 = 0\} = \{0\}$, the origin of $\mathbb{C}^3$; from above: $\sigma_1 = \Sigma \cap \{y_1 > 0\}$; $\sigma_2 = \Sigma \cap \{y_1 < 0\}$.

(*ii*) The three connected components of $S \setminus \Sigma'$ are the components which contain, respectively e_1, e_2, e_3 and whose boundaries are $\overline{\sigma}_1$, $\overline{\sigma}_2$, $\overline{\sigma}_1 \cup \overline{\sigma}_2$; these boundaries have corners as shown in the first part of the proof. $\square$

The connected component of $\mathbb{C}^2 \times \mathbb{R} \setminus S$ containing the point $(0, 0, 0, 0, 1/2)$ is the Levi-flat solution, the complex leaves being the sections by the hyperplanes $x_3 = x_3^0$, $-1 < x_3^0 < 1$.

The sections by the hyperplanes $x_3 = x_3^0$ are diffeomorphic to a 3-sphere for $0 < x_3^0 < 1$ and to the union of two disjoint 3-spheres for $-1 < x_3^0 < 0$, as can be shown intersecting S by lines through the origin in the hyperplane $x_3 = x_3^0$; Σ' is homeomorphic to the union of two 3-spheres with a common point.

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3.4. Sphere with one special 1-hyperbolic point (sphere with two horns). The example of section 3.3 shows that the necessary conditions of section 2 can be realised. Moreover, from Proposition 2.8.7, the hypothesis on the number of complex points is meaningful.

3.4.1.

Proposition 14. *[cf* [Dol08]*[Proposition 2.6.1]] Let $S \subset \mathbb{C}^n$ be a compact connected real 2-codimensional manifold such that the following holds:*
(i) S is a topological sphere; S is nonminimal at every CR point;
(ii) every complex point of S is flat; there exist three special elliptic points $e_j, j = 1, 2, 3$ and one special 1-hyperbolic point h;
(iii) S does not contain complex manifolds of dimension $(n-2)$;
(iv) the singular CR *orbit Σ' through h on S is compact and $\Sigma' \setminus \{h\}$ has two connected components σ_1 and σ_2 whose closures are homeomorphic to spheres of dimension $2n-3$;*
(v) the closures S_1, S_2, S_3 of the three connected components S_1', S_2', S_3' of $S \setminus \Sigma'$ are submanifolds with (singular) boundary.
Then each $S_j \setminus \{e_j \cup \Sigma'\}$, $j = 1, 2, 3$ carries a foliation $\mathcal{F}_j$ of class C^∞ with 1-codimensional CR *orbits as compact leaves.*

Proof. From conditions (i) and (ii), S satisfying the hypotheses of Proposition 1, near any elliptic flat point e_j, and of Proposition 6 near Σ', all CR orbits being diffeomorphic to the sphere $\mathbf{S}^{2n-3}$. The assumption (iii) guarantees that all CR orbits in S must be of real dimension $2n-3$. Hence, by removing small connected open saturated neighborhoods of all special elliptic points, and of Σ', we obtain, from $S \setminus \Sigma'$, three compact manifolds S_j", $j = 1, 2, 3$, with boundary and with the foliation $\mathcal{F}_j$ of codimension 1 given by its CR orbits whose first cohomology group with values in $\mathbf{R}$ is 0, near e_j. It is easy to show that this foliation is transversely oriented. □

3.4.2. Recall the Thurston's Stability Theorem ([CaC], Theorem 6.2.1).

Proposition 15. *Let $(M, \mathcal{F})$ be a compact, connected, transversely-orientable, foliated manifold with boundary or corners, of codimension 1, of class C^1. If there is a compact leaf L with $H^1(L, \mathbf{R}) = 0$, then every leaf is homeomorphic to L and M is homeomorphic to $L \times [0, 1]$, foliated as a product,*

Then, from the above theorem, S_j" is homeomorphic to $\mathbf{S}^{2n-3} \times [0, 1]$ with CR orbits being of the form $\mathbf{S}^{2n-3} \times \{x\}$ for $x \in [0, 1]$. Then the full manifold S_j is homeomorphic to a half-sphere supported by $\mathbf{S}^{2n-2}$ and $\mathcal{F}_j$ extends to S_j; S_3 having its boundary pinched at the point h.

□

3.4.3.

Theorem 16. *Let $S \subset \mathbb{C}^n$, $n \geq 3$, be a compact connected smooth real 2-codimensional submanifold satisfying the conditions (i) to (v) of Proposition 15. Then there exists a Levi-flat $(2n-1)$-subvariety $\tilde{M} \subset \mathbb{C} \times \mathbb{C}^n$ with boundary $\tilde{S}$ (in the sense of currents) such that the natural projection π :*

$\mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ *restricts to a bijection which is a CR diffeomorphism between* $\tilde{S}$ *and* S *outside the complex points of* S.

Proof. By Proposition 1, for every e_j, a continuous function ν'_j, C^∞ outside e_j, can be constructed in a neighborhood U_j of e_j, $j = 1, 2, 3$, and by Proposition 6, we have an analogous result in a neighborhood of Σ'. Furthermore, from Proposition 15, a smooth function $\nu"_j$ whose level sets are the leaves of $\mathcal{F}_j$ can be obtained globally on $S'_j \setminus \{e_j \cup \Sigma'\}$. With the functions ν'_j and $\nu"_j$, and analogous functions near Σ', then using a partition of unity, we obtain a global smooth function $\nu_j \colon S_j \to \mathbf{R}$ without critical points away from the complex points e_j and from Σ'.

Let σ_1, resp. σ_2 be the two connected, relatively compact components of $\Sigma \setminus \{h\}$, according to condition (iv); $\overline{\sigma}_1$, resp. $\overline{\sigma}_2$ are the boundary of S_1, resp. S_2, and $\overline{\sigma}_1 \cup \overline{\sigma}_2$ the boundary of S_3. We can assume that the three functions ν_j are finite valued and get the same values on $\overline{\sigma}_1$ and $\overline{\sigma}_2$. Hence a function $\nu : S \to \mathbf{R}$.

The submanifold S being, locally, a boundary of a Levi-flat hypersurface, is orientable. We now set $\tilde{S} = N = \operatorname{gr} \nu = \{(\nu(z), z) : z \in S\}$. Let $S_s = \{e_1, e_2, e_3, \overline{\sigma_1 \cup \sigma_2}\}$.

$\lambda : S \to \tilde{S}$ $\big(z \mapsto \nu((z), z)\big)$ is bicontinuous; $\lambda|_{S \setminus S_s}$ is a diffeomorphism; moreover λ is a CR map. Choose an orientation on S. Then N is an (oriented) CR subvariety with the negligible set of singularities $\tau = \lambda(S_s)$.

At every point of $S \setminus S_s$, $d_{x_1}\nu \neq 0$, then condition (H) (section 3.1.1) is satisfied at every point of $N \setminus \tau$.

Then all the assumptions of Theorem 10 being satisfied by $N = \tilde{S}$, in a particular case, we conclude that N is the boundary of a Levi-flat $(2n-2)$-variety (with negligible singularities) $\tilde{M}$ in $\mathbf{R} \times \mathbb{C}^n$.

Taking $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ to be the standard projection, we obtain the conclusion. □

3.5. Generalizations: elementary models and their gluing.

3.5.1. The examples and the proofs of the theorems when S is homeomorphic to a sphere (sections 3.4) suggest the following definitions.

3.5.2. *Definitions.* Let T' be a smooth, locally closed (i.e. closed in an open set), connected submanifold of $\mathbb{C}^n$, $n \geq 3$. We assume that T' has the following properties:

(i) T' is relatively compact, non necessarily compact, and of codimension 2.

(ii) T' is nonminimal at every CR point.

(iii) T' does not contain complex manifold of dimension $(n-2)$.

(iv) T' has exactly 2 complex points which are flat and either special elliptic or special 1-hyperbolic.

(v) If $p \in T'$ is special 1-hyperbolic, the singular orbit Σ' through p is compact, $\Sigma' \setminus p$ has two connected components σ_1, σ_2, whose closures are homeomorphic to spheres of dimension $2n-3$.

(vi) If $p \in T'$ is special 1-hyperbolic, in the neighborhood of p, with convenient coordinates, the equation of T', up to third order terms is

$$z_n = \sum_{j=1}^{n-1} (z_j \overline{z}_j + \lambda_j \mathcal{R}e\ z_j^2);\ \lambda_1 > 1;\ 0 \leq \lambda_j < 1 \ \text{ for } \ j \neq 1$$

or in real coordinates x_j, y_j with $z_j = x_j + iy_j$,

$$x_n = \big((\lambda_1+1)x_1^2 - (\lambda_1-1)y_1^2\big) + \sum_{j=2}^{n-1} \big((1+\lambda_j)x_j^2 + (1-\lambda_j)y_j^2\big) + O(|z|^3)$$

(vii) the closures, in T', T_1, T_2, T_3 of the three connected components T_1', T_2', T_3' of $T' \setminus \Sigma'$ are submanifolds with (singular) boundary. Let $T"_j$, $j = 1, 2, 3$ be neighborhoods of the T_j' in T'.

up- and down- 1*-hyperbolic points.* Let τ be the $(2n-2)$-submanifold with (singular) boundary contained into T' such that either $\overline{\sigma}_1$ (resp. $\overline{\sigma}_2$) is the boundary of τ near p, or Σ' is the boundary of τ near p. In the first case, we say that p is 1-*up*, (resp. 2-*up*), in the second that p is *down*. If T' is contained in a small enough neighborhood of Σ' in $\mathbb{C}^n$, such a T' will be called a *local elementary model*, more precisely it defines a *germ of elementary model around* Σ.

The union T of T_1, T_2, T_3 and of the germ of elementary model around the singular orbit at every special 1-hyperbolic point is called an *elementary model*. T behaves as a locally closed submanifold still denoted T.

3.5.3. *Examples of elementary models.* We will say that T is a *elementary model of type*:
(a) if it has: two elliptic points;
(b) if it has: one special elliptic point and one *down*-$\{1\}$-hyperbolic point;
(c_1) if it has: one special elliptic point and one 1-*up*-$\{1\}$-hyperbolic point;
(c_2) if it has: one special elliptic point and one 2-*up*-$\{1\}$-hyperbolic point;
(d_1) if it has: two special 1-*up*-$\{1\}$-hyperbolic points;
(d_2) if it has: two special 2-*up*-$\{1\}$-hyperbolic points;
(e) if it has: two special *down*-$\{1\}$-hyperbolic points;

Other configurations are easily imagined.

The prescribed boundary of a Levi-flat hypersurface of $\mathbb{C}^n$ in [DTZ05] and [DTZ10], whose complex points are flat and elliptic, is an elementary model of type (a).

3.5.4. *Properties of elementary models.* For instance, T is 1-up and has one special elliptic point, we solve the boundary problem as in S_1 in the proof of Theorem 16.

Proposition 17. *Let T be a local elementary model. Then, T carries a foliation $\mathcal{F}$ of class C^∞ with* 1*-codimensional* CR *orbits as compact leaves.*

Proof. From the definition at the end of section 3.5.2 and Proposition 6. □

3.5.5.

Theorem 18. *Let T be the elementary model there exists an open neighborhood T" in T' carrying a smooth function $\nu : T" \to \mathbb{R}$ whose level sets are the leaves of a smooth foliation.*

Proof. By removing small connected open saturated neighborhoods of every special elliptic point, and of Σ', the singular orbit through every special 1-hyperbolic point p, we obtain, from $S \setminus \Sigma'$, three compact manifolds S_j", $j = 1, 2, 3$, with boundary,

(a) S_1 and S_2 containing one special elliptic point e or one special 1-hyperbolic point with the foliations $\mathcal{F}_1$, $\mathcal{F}_2$, from Propositions 1 and 17,

(b) S_3" with the foliation $\mathcal{F}_3$ of codimension 1 given by its CR orbits whose first cohomology group with values in $\mathbf{R}$ is 0, near e, or p. It is easy to show that this later foliation is transversely oriented.

From the Thurston's Stability Theorem (see section 3.4.2), S_3" is homeomorphic to $\mathbf{S}^{2n-3} \times [0, 1]$, foliated as a product, with CR orbits being of the form $\mathbf{S}^{2n-3} \times \{x\}$ for $x \in [0, 1]$; hence smooth functions ν_1, ν_2, ν_3, whose level sets are the leaves of the foliations $\mathcal{F}_1$, $\mathcal{F}_2, \mathcal{F}_3$ respectively, and using a partition of unity the desired function ν on T.

□

3.6.

Theorem 19. *Let T be an elementary model. Then there exists a Levi-flat $(2n-1)$-subvariety $\tilde{M} \subset \mathbb{C} \times \mathbb{C}^n$ with boundary $\tilde{T}$ (in the sense of currents) such that the natural projection $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ restricts to a bijection which is a CR diffeomorphism between $\tilde{T}$ and T outside the complex points of T.*

Proof. The submanifold T being, locally, a boundary of a Levi-flat hypersurface, is orientable. We now set $\tilde{T} = N = \mathrm{gr}\,\nu = \{(\nu(z), z) : z \in S\} \subset E \cong \mathbf{R} \times \mathbb{C}^{n-1}$. Let T_s be the union of the flat complex points of T.

$\lambda : T \to \tilde{T}$ $\big(z \mapsto \nu((z), z)\big)$ is bicontinuous; $\lambda|_{T \setminus T_s}$ is a diffeomorphism; moreover λ is a CR map. Choose an orientation on T. Then N is an (oriented) CR subvariety with the negligible set of singularities $\tau = \lambda(T_s)$.

Using Remark 11, at every point of $T \setminus T_s$, $d_{x_1}\nu \neq 0$, we see that condition (H) (section 3.1.1) is satisfied at every point of $N \setminus \tau$.

Then all the assumptions of Theorem 10 being satisfied by $N = \tilde{T}$, in a particular case, we conclude that N is the boundary of a Levi-flat $(2n-2)$-variety (with negligible singularities) $\tilde{M}$ in $\mathbf{R} \times \mathbb{C}^n$.

Taking $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ to be the standard projection, we obtain the conclusion. □

3.7. Gluing of elementary models.

3.7.1. The gluing happens between two compatible elementary models along boundaries, for instance down and 1-up. Remark that the gluing can only be made at special 1-hyperbolic points. More precisely, it can be defined as follows.

The assumed properties of the submanifold S in section 2 in $\mathbb{C}^n$ have a meaning in any complex analytic manifold X of complex dimension $n \geq 3$, and are kept under any holomorphic isomorphism.

We will define a submanifold S' of X obtained by gluying of elementary models by induction on the number m of models. An elementary model T in X is the image of an elementary model T_0 in $\mathbb{C}^n$ by an analytic isomorphism of a neighborhood of T_0 in $\mathbb{C}^n$ into X.

3.7.2. Let S' be a closed smooth real submanifold of X of dimension $2n-2$ which is non minimal at every CR point. Assume that S' is obtained by gluing of m elementary models.

a) S' has a finite number of flat complex points, some special elliptic and the others special 1-hyperbolic;

b) for every special 1-hyperbolic p', there exists a CR-isomorphism h induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ from a neighborhood of p in T' onto a neighborhood of p' in S'.

c) for every CR-orbit $\Sigma_{p'}$ whose closure contains a special 1-hyperbolic point p', there exists a CR-isomorphism h induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ from a neighborhood of $\Sigma_p = \Sigma'_p \setminus p$ in T' onto a neighborhood V of $\Sigma_{p'}$ in S'.

Every special 1-hyperbolic point of S' which belongs to only one elementary model in S' will be called *free*.

We will define the gluing of one more elementary model to S'.

3.7.3. *Gluing an elementary model T of type (d_1) to a free down-1-hyperbolic point of S'.* Let h_1 be a CR-isomorphism from a neighborhood V_1 of $\overline{\sigma}'_1$ induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ onto a neighborhood of σ_1 in S'. Let k_1 be a CR-isomorphism from a neighborhood $T"_1$ of T'_1 into X such that $k_1|V_1 = h_1$.

3.7.4.

Theorem 20. *The compact manifold or the manifold with singular boundary S', obtained by the gluing of a finite number of elementary models, is the boundary of a Levi-flat hypersurface of X in the sense of currents.*

Proof. From Theorem 19 and the definition of gluing. $\square$

3.8. Examples of gluing. Denoting the gluing of the two models of type (d_1) and (d_2) to a free down-1-hyperbolic point of S' by: $\rightarrow (d_1) - (d_2)$, and the converse by: $(d_1) - (d_2) \rightarrow$, and, also, analogous configurations in the same way, we get:

torus: $(b) \rightarrow (d_1) - (d_2) \rightarrow (b)$; the Euler-Poincaré characteristic of a torus is $\chi(\mathbf{T}^k) = 0$: 2 special elliptic and 2 special 1-hyperbolic points.

bitorus: $(b) \rightarrow (d_1) - (d_2) \rightarrow (e) \rightarrow (d_1) - (d_2) \rightarrow (b)$.

4. Case of graphs

(see [DTZ09] for the case of elliptic points only, and dropping the property of the function solution to be Lipschitz).

4.1. We want to add the following hypothesis: S is embedded into the boundary of a strictly pseudoconvex domain of $\mathbb{C}^n$, $n \geq 3$, and more precisely, let (z, w) be the coordinates in $\mathbb{C}^{n-1} \times \mathbb{C}$, with $z = (z_1, \ldots, z_{n-1})$, $w = u + iv = z_n$, let Ω be a strictly pseudoconvex domain of $\mathbb{C}^{n-1} \times \mathbb{R}_u$ (i.e. the second fundamental form of the boundary $b\Omega$ of Ω is everywhere positive definite); let S be the graph $gr(g)$ of a smooth function $g : b\Omega \rightarrow \mathbb{R}_v$. notice that $b\Omega \times \mathbb{R}_v$ contains S and is strictly pseudoconvex.

Assume that S is a *horned sphere* (section 3.4), *satisfying the hypotheses of* Theorem 16. Denote by p_j, $j = i, \ldots, 4$ the complex points of S. Our aim is to prove

4.2.

Theorem 21. *Let S be the graph of a smooth function $g : b\Omega \rightarrow \mathbb{R}_v$. Let $Q = (q_1, \ldots, q_4) \in b\Omega$ be the projections of the complex points $P = (p_1, \ldots, p_4)$ of S, respectively. Then, there exists a continuous function $f : \overline{\Omega} \rightarrow \mathbb{R}_v$ which is smooth on $\overline{\Omega} \setminus Q$ and such that $f_{|b\Omega} = g$, and $M_0 = graph(f) \setminus S$ is a smooth Levi flat hypersurface of $\mathbb{C}^n$. Moreover, each complex leaf of M_0 is the graph of a holomorphic function $\phi : \Omega' \rightarrow \mathbb{C}$ where $\Omega' \subset \mathbb{C}^{n-1}$ is a domain with smooth boundary (that depends on the leaf) and ϕ is smooth on $\overline{\Omega}'$.*

The natural candidate to be the graph M of f is $\pi(\tilde{M})$ where $\tilde{M}$ and π are as in Theorem 16. We prove that this is the case proceeding in several steps.

4.3. Behaviour near S.

4.3.1. *Assume that D is a strictly pseudoconvex domain and that $S \subset bD$.*

Recall ([HL75][Theorem 10.4]: *Let D be a strictly pseudoconvex domain of $\mathbb{C}^n$, $n \geq 3$ with boundary bD, $\Sigma \subset bD$ be a compact connected maximally complex smooth $(2d-1)$-submanifold with $d \geq 2$. Then, Σ is the boundary of a uniquely determined relatively compact subset $V \subset \overline{D}$ such that $\overline{V} \setminus \Sigma$ is a complex analytic subset of D with finitely many singularities of pure* *dimension* $\leq d-1$, *and near* Σ, $\overline{V}$ *is a* d*-dimensional complex manifold with boundary.*

V is said to be *the solution of the boundary problem for* Σ.

4.3.2.

Lemma 22 ([DTZ09])**.** *Let* Σ_1, Σ_2 *be compact connected maximally complex* $(2d-1)$*-submanifolds of* bD. *Let* V_1, V_2 *be the corresponding solutions of the boundary problem. If* $d \geq 2$, $2d \geq n+1$ *and* $\Sigma_1 \cap \Sigma_2 = \emptyset$, *then* $V_1 \cap V_2 = \emptyset$.

Let Σ be a CR orbit of the foliation of $S \setminus P$. Then Σ is a compact maximally complex $(2n-3)$-dimensional real submanifold of $\mathbb{C}^n$ contained in bD. Let $V = V_\Sigma$ be the solution of the boundary problem corresponding to Σ. From Theorem 16, $V = \pi(\tilde{V})$, where $\tilde{V} = (\tilde{M} \setminus \tilde{S}) \cap (\mathbb{C}^n \times \{x\})$ for suitable $x \in (0,1)$, the projection on the x-axis being finite, we can always assume that it lies into $(0,1)$. Moreover $\pi_{|\tilde{V}}$ is a biholomorphism $\tilde{V} \cong V$ and $M \setminus S \subset D$.

Let Σ_1, Σ_2 be two distinct orbits of the foliation of $S \setminus P$, and $\overline{V}_1, \overline{V}_2$ the corresponding leaves, then, from Lemma 22, $\overline{V}_1 \cap \overline{V}_2 = \emptyset$.

4.3.3. *Assume that* S *satisfies the full hypotheses of* Theorem 21.

Set $m_1 = \min_S g$, $m_2 = \max_S g$ and $r \gg 0$ such that

$$D = \Omega \times [m_1, m_2] \subset\subset \mathbf{B}(\mathbf{r}) \cap \big(\Omega \times i\mathbb{R}_v\big)$$

where $\mathbf{B}(\mathbf{r})$ is the ball $\{|(z,w)| < r\}$.

4.3.4.

Lemma 23. *Let* $p \in S$ *be a CR point. Then, near* p, M *is the graph of a function* ϕ *on a domain* $U \subset \mathbb{C}_z^{n-1} \times \mathbb{R}_u$ *which is smooth up to the boundary of* U.

Proof. Near p, each CR orbit Σ is smooth and can be represented as the graph of a CR function over a strictly pseudoconvex hypersurface and V_Σ as the graph of the local holomorphic extension of this function. From Hopf lemma, V is transversal to the strictly pseudoconvex hypersurface $d\Omega \times i\mathbb{R}_v$ near p. Hence the family of the V_Σ, near p, forms a smooth real hypersurface with boundary on S that is the graph of a smooth function ϕ from a relative open neighborhood U of p on $\overline{\Omega}$ into $\mathbb{R}_v$. Finally, Lemma 22 garantees that this family does not intersect any other leaf V from M. $\square$

4.3.5.

Corollary 24. *If* $p \in S$ *is a CR point, each complex leaf* V *of* M, *near* p, *is the graph of a holomorphic function on a domain* $\Omega_V \subset \mathbb{C}_z^{n-1}$, *which is smooth up to the boundary of* Ω_V.

4.4. Solution as a graph of a continuous function.

4.4.1. Recall results of Shcherbina [Shc93] from:

(a) the Main Theorem:

Let G be a bounded strictly convex domain in $\mathbb{C}_z \times \mathbb{R}_u$ ($z \in \mathbb{C}$) and $\varphi : bG \to \mathbb{R}_v$ be a continuous function. Then the following properties hold, where $\Gamma = gr$, and $\hat{\Gamma}(\varphi)$ means polynomial hull of $\Gamma(\varphi)$:

(a_i) the set $\hat{\Gamma}(\varphi) \setminus \Gamma(\varphi)$ is the union of a disjoint family of complex discs $\{D_\alpha\}$;

(a_{ii}) for each α, there is a simply connected domain $\Omega_\alpha \subset \mathbb{C}_z$ and a holomorphic function $w = f_\alpha$, defined on Ω_α, such that D_α is the graph of f_α.

(a_{iii}) For each f_α, there exists an extension $f^*_\alpha \in C(\overline{\Omega}_\alpha)$ and $bD_\alpha = \{(z,w) \in b\Omega_\alpha \times \mathbb{C}_w : w = f^*_\alpha(z)\}$.

(b)

Lemma 25. *Let $\{G_n\}_{n=0}^\infty$, $G_n \subset \mathbb{C}_z \times \mathbb{R}_u$, be a sequence of bounded strictly convex domains such that $G_n \to G_0$. Let $\{\varphi_n\}_{n=0}^\infty$, $\varphi_n : \partial G_n \to \mathbb{R}_v$ be a sequence of continuous functions such that $\Gamma(\varphi_n) \to \Gamma(\varphi_0)$ in the Hausdorff metric. Then, if Φ_n is the continuous function $: \overline{G}_n \to \mathbb{R}_v$ such that $\hat{\Gamma}(\varphi) = \Gamma(\Phi)$, we have $\Gamma(\Phi_n) \to \Gamma(\Phi_0)$ in the Hausdorff metric.*

(c)

Lemma 26. *Let $\mathcal{U}$ be a smooth connected surface which is properly embedded into some convex domain $G \subset \mathbb{C}_z \times \mathbb{R}_u$. Suppose that near each point of this surface, it can be defined locally by the equation $u = u(z)$. Then the surface $\mathcal{U}$ can be represented globally as a graph of some function $u = U(z)$, defined on some domain $\Omega \subset \mathbb{C}_z$.*

4.4.2.

Proposition 27. *M is the graph of a continuous function $f : \overline{\Omega} \to \mathbb{R}_v$.*

Proof. We will intersect the graph S with a convenient affine subspace of real dimension 4 to go back to the situation of Shcherbina.

Fix $a \in (\mathbb{C}_z^{n-1} \setminus 0)$ and, for a given point $(\zeta, \xi) \in \Omega$, with $\zeta \in \mathbb{C}_z^{n-1}$ and $\xi \in \mathbb{R}_u$, let $H_{(\zeta,\xi)} \subset \mathbb{C}_z^{n-1} \times \{\xi\}$ be the complex line through (ζ, ξ) in the direction $(a, 0)$. Set:

$$L_{(\zeta,\xi)} = H_{(\zeta,\xi)} + \mathbb{R}_u(0,1), \quad \Omega_{(\zeta,\xi)} = L_{(\zeta,\xi)} \cap \Omega, \quad S_{(\zeta,\xi)} = (H_{(\zeta,\xi)} + \mathbb{C}_w(0,1)) \cap S$$

Then $S_{(\zeta,\xi)}$ is contained in the strictly convex cylinder

$$(H_{(\zeta,\xi)} + \mathbb{C}_w(0,1)) \cap (b\Omega \times i\mathbb{R}_v)$$

and is the graph of $g_{|b\Omega_{(\zeta,\xi)}}$.

From (a_{ii}), the polynomial hull of $S_{(\zeta,\xi)}$ is a continuous graph over $\overline{\Omega}_{(\zeta,\xi)}$. Consider $M = \pi(\tilde{M})$ and set

$$M_{\zeta,\xi)} = (H_{(\zeta,\xi)} + \mathbb{C}_w(0,1)) \cap M.$$

It follows that $M_{\zeta,\xi)}$ is contained in the polynomial hull $\hat{S}_{(\zeta,\xi)}$. From (a_{iii}), $\hat{S}_{(\zeta,\xi)}$ is a graph over $\overline{\Omega}_{(\zeta,\xi)}$ foliated by analytic discs, so $M_{\zeta,\xi)}$ is a graph over a subset U of $\overline{\Omega}_{(\zeta,\xi)}$.

Every analytic disc Δ of $\hat{S}_{(\zeta,\xi)}$ had its boundary on $S_{(\zeta,\xi)}$. Since all the the complex points of S are isolated, $b\Delta$ contains a CR point p of S; from Lemma 23, near p, $M_{\zeta,\xi)}$ is a graph over $\overline{\Omega}_{(\zeta,\xi)}$. Near p, Δ is contained in $M_{\zeta,\xi)}$, then in a closed complex analytic leaf V_Σ of M; so $\Delta \subset V_\Sigma \subset M$; but $\Delta \subset H_{(\zeta,\xi)}+\mathbb{C}_w(0,1)$; then: $\Delta \subset M_{\zeta,\xi)}$. Consequently, near p, $M_{\zeta,\xi)} = \hat{S}_{(\zeta,\xi)}$.

It follows that M is the graph of a function $f : \overline{\Omega} \to \mathbb{R}_v$.

One proves, using (b), that f is continuous on Ω, whence on $\overline{\Omega} \setminus Q$, by Lemma 23. Then continuity at every q_j is proved using the Kontinuitätsatz on the domain of holomorphy $\Omega \times i\mathbb{R}_v$. □

4.5. Regularity. The property: $M \setminus P = (p_1, \ldots, p_4)$ *is a smooth manifold with boundary* results from:

4.5.1.

Lemma 28. *Let U be a domain of $\mathbb{C}_z^{n-i} \times \mathbb{R}_u, n \geq 2, f : U \to \mathbb{R}_v$ a continuous function. Let $A \subset \mathrm{graph}(f)$ be a germ of complex analytic set of codimension 1. Then A is a germ of complex manifold which is a graph of over $\mathbb{C}_z^{n-i}$.*

Proof. Assume that A is a germ at 0. Let $g \in \mathcal{O}, h \neq 0$ such that $A = \{h = 0\}$. For $\varepsilon << 1$, let $\mathbf{D}_\varepsilon$ be the disc $\{z = 0\} \cap \{|w| < \varepsilon\}$, then $A \cap \mathbf{D}_\varepsilon = \{0\}$, i.e. A is w-regular.

Let $\pi : \mathbb{C}^n_{z,w} \to \mathbb{C}_z^{n-1}$ be the projection. The local structure theorem for analytic sets gives:

for some neighborhood U of 0 in $\mathbb{C}_z^{n-1}$, there exists an analytic hypersurface $\Delta \subset U$ such that: $A_\Delta = A \cup ((U \setminus \Delta) \times \mathbf{D}_\varepsilon)$ is a manifold;

$\pi/A_\Delta \to U \setminus \Delta$ is a $d(\in \mathbb{N})$-sheeted covering.

It is easy to show that the covering $\pi : A_\Delta \to U \setminus \Delta$ is trivial.

Then we may define d holomorphic functioons $\tau_1, \ldots, \tau_d : U \setminus \Delta \to \mathbb{C}$ such that A_Δ is the union of the graphs of the τ_j. By the Riemann extension theorem, the functions τ_j extend as holomorphic functions $\tau_j \in \mathcal{O}(U)$. Suppose that $\tau_j \neq \tau_k$, for $j \neq k$, then for some disc $\mathbf{D} \subset U$ centered at 0, we have $\tau_j|\mathbf{D} \neq \tau_k|\mathbf{D}$, then $(\tau_j - \tau_k)|_{\mathbf{D}}$ vanishes only at 0. But, from the hypothesis, in restriction to $\mathbf{D}$, $\{Re(\tau_j - \tau_k) = 0\} \subset \{\tau_j - \tau_k = 0\}|_{\mathbf{D}} = \{0\}$, impossible. □

4.6.

Proof of the Theorem 21. Consider the foliation of $S \setminus P$ given by the level sets of the smooth function $\nu : S \to [0, 1]$ (sections 2.3 and 2.7) and set $L_t = \{\nu = t\}$ for $t \in (0.1)$. Let $V_t \subset \overline{\Omega} \times i\mathbb{R}_v \subset \mathbb{C}^n$ be the complex leaf of M bounded by L_t.

By Proposition 27, M is the graph of a continuous function over Ω, and, by Lemma 28, each leaf V_t is a complex smooth hypersurface and $\pi|_{V_t}$ is a submersion.

$\rightarrow$

Since Ω is strictly convex, as in Shcherbina (see 4.4.1, c)), $\pi_{|V_t}$ is 1-1, then, by Corollary 24, π sends V_t onto a domain $\Omega_t \subset \mathbb{C}_z^{n-1}$ with smooth boundary. Let

$\pi_u : (\mathbb{C}_z^{n-1} \times \mathbb{R}_u) \times i\mathbb{R}_v \rightarrow \mathbb{R}_u$

$\pi_v : (\mathbb{C}_z^{n-1} \times \mathbb{R}_u) \times i\mathbb{R}_v \rightarrow \mathbb{R}_v$

then $\pi_{u|L_t} = a_t.\pi_{|L_t}$ and $\pi_{v|L_t} = b_t.\pi_{|L_t}$ where a_t, b_t are smooth functions on $b\Omega_t$. Moreover $b\Omega_t$, a_t, b_t depend smoothly on t.

If $(z_t, w_t) \in M$, then w_t varies on V_t, so w_t is the holomorphic extension of $a_t + ib_t$ to Ω_t. In particular u_t and v_t are smooth in (z,t), from the Bochner-Martinelli formula.

$\dfrac{\partial u_t}{\partial t}$ is harmonic on Ω_t for each t and has a smooth extension on $b\Omega_t$.

From Lemma 23 and Corollary 24, $\dfrac{\partial u_t}{\partial t}$ does not vanish on $b\Omega_t$. Since the CR orbits L_t are connected from Proposition 14, $b\Omega_t$ is also connected, hence $\dfrac{\partial u_t}{\partial t}$ has constant sign on $b\Omega_t$. Then, by the maximum principle, also $\dfrac{\partial u_t}{\partial t}$ on Ω_t and, in particular does not vanish. This implies that $M \setminus S$ is the graph of a smooth function over Ω which smoothly extends to $\overline{\Omega} \setminus Q$.

From Proposition 27, M is the graph of a continuous function over $\overline{\Omega}$. $\square$

4.7. Elementary smooth models.

4.7.1. *Definition. An elementary smooth model in* $\mathbb{C}^n$ is an elementary model in the sense of section 3.5.2 and satisfying the further condition which makes sense from Theorem 21:

(G) Let (z, w) be the coordinates in $\mathbb{C}^{n-1} \times \mathbb{C}$, with $z = (z_1, \ldots, z_{n-1})$, $w = u + iv = z_n$, let Ω be a strictly pseudoconvex domain of $\mathbb{C}^{n-1} \times \mathbb{R}_u$; assume that T' is the graph of a smooth function $g : b\Omega \rightarrow \mathbb{R}_v$.

4.7.2.

Theorem 29. *Let T be an elementary smooth model. Then, there exists a continuous function $f : \overline{\Omega} \rightarrow \mathbb{R}_v$ which is smooth on $\overline{\Omega} \setminus Q$ and such that $f_{|b\Omega} = g$, and $M_0 = graph(f) \setminus S$ is a smooth Levi flat hypersurface of $\mathbb{C}^n$; in particular, S is the boundary of the hypersurface $M = graph(f)$*

Proof. similar to the proof of Theorem 21. $\square$

4.7.3. *Gluing of elementary smooth models.* In an open set of $\mathbb{C}^n$, a coordinate system (z, w) of $\mathbb{C}_z^{n-1} \times \mathbb{R}_u$ defines an $(n-1, 1)$-*frame.*

To define the gluing of elementary models (section 3.7) we considered a CR-isomorphism from an open set of $\mathbb{C}^n$ induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ onto a an open set of $\mathbb{C}^n$. To define the gluing of elementary smooth models, we have to consider a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ onto an open set of $\mathbb{C}^n$ sending an $(n-1,1)$-frame of $\mathbb{C}^{n-1}_z \times \mathbb{R}_u$ onto an $(n-1,1)$-frame of $\mathbb{C}^{n-1}_{z'} \times \mathbb{R}_{u'}$.

As in section 3.7.1, we will define a submanifold S' of X obtained by gluing of elementary smooth models by induction on the number m of models. An elementary smooth model T in X is the image of an elementary smooth model T_0 of $\mathbb{C}^n$ by an analytic isomorphism of a neighborhood of T_0 in $\mathbb{C}^n$ into X.

Gluing an elementary smooth model T of type (d_1) to a free down-1-hyperbolic point of S'.

Every elementary smooth model is contained in a cylinder $b\Omega \times \mathbb{R}_v$ determined by Ω and an $(n-1,1)$-frame. Two sets Ω are *compatible* if either they coincide or one is part of the other.

The announced gluing is defined in the following way: there exists a CR-isomorphism h_1 from a neighborhood V_1 of $\overline{\sigma}'_1$ induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ onto a neighborhood of σ_1 in S'. Let k_1 be a CR-isomorphism from a neighborhood $T"_1$ of T'_1 into X such that $k_1|V_1 = h_1$, and there exists a common $(n-1,1)$-frame on which the corresponding sets Ω are compatible. The existence of such a situation is possible as the example of the horned (almost everywhere) smooth sphere shows (Theorem 21.).

Remark that the gluing implies that the obtained submanifold S' is C^0 and smooth except at the complex points.

Other gluing are obtained in a similar way. Hence:

Theorem 30. *The manifold S' obtained by gluing of elementary smooth models is of class C^0, and smooth except at the complex points.*

Corollary 31. *The manifold S' is the boundary of manifold M of class C^∞ whose interior is a Levi-flat smooth hypersurface.*

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Institut de Mathématiques de Jussieu, UPMC, 4, place Jussieu 75005 Paris
E-mail address: pierre.dolbeault@upmc.fr